$$\Rightarrow v_{i1x} = \frac{\frac{ek\phi_1}{m_i\omega}}{1 - \frac{\Omega_i^2}{\omega^2}} \quad \text{--- (13)}$$

changing e to $-e$ and m_i to m_e
(ion) (electron)

and Ω_i to $-\Omega_e$ in (13)

we can write the above result in the linearised form of equations for electron with $KT_e = 0$ & we have

$$v_{e1x} = \frac{-\frac{ek\phi_1}{m_e\omega}}{1 - \frac{\Omega_e^2}{\omega^2}} \quad \text{--- (14)}$$

now the equations of ~~for~~ of continuity for ions and electron are

$$\frac{\partial n_e}{\partial t} + \vec{\nabla}\cdot(n_e v_e) = 0$$

$$\frac{\partial n_i}{\partial t} + \vec{\nabla}\cdot(n_i \cdot v_i) = 0$$

using linearisation and plane wave solution for the above two equations we have

$$-i\omega n_{e1} + n_0(ik)v_{e1x} = 0$$

$$-i\omega n_{i1} + n_0(ik)v_{i1x} = 0$$

$$n_{e1} = \frac{n_0 k}{\omega} v_{e1x} \quad \text{--- (15)}$$

$$n_{i1} = \frac{n_0 k}{\omega} v_{i1x} \quad \text{--- (16)}$$

using the plasma approximations that is $n_{i1} = n_{e1}$ we get

~~xx~~ using (15) & (16) we have

$$v_{e1x} = v_{i1x}$$

$$\frac{\dfrac{-ek\phi_1}{m_e\omega}}{1-\dfrac{\Omega_e^2}{\omega^2}} = \frac{\dfrac{ek\phi_1}{m_i\omega}}{1-\dfrac{\Omega_i^2}{\omega^2}}$$

$$\Rightarrow \left(-\frac{ek\phi_1}{m_e\omega}\right)\left(1-\frac{\Omega_e^2}{\omega^2}\right)^{-1} = \left(\frac{ek\phi_1}{m_i\omega}\right)\left(1-\frac{\Omega_i^2}{\omega^2}\right)^{-1}$$

$$\Rightarrow -\frac{1}{m_e}\left[1-\frac{\Omega_e^2}{\omega^2}\right]^{-1} = \frac{1}{m_i}\left[1-\frac{\Omega_i^2}{\omega^2}\right]^{-1}$$

$$\Rightarrow m_e\left(1-\frac{\Omega_e^2}{\omega^2}\right) = -m_i\left(1-\frac{\Omega_i^2}{\omega^2}\right)$$

$$\Rightarrow m_e + m_i = m_i\frac{\Omega_i^2}{\omega^2} + m_e\frac{\Omega_e^2}{\omega^2}$$

$$\Rightarrow (m_e+m_i)\,\omega^2 = m_i\,\Omega_i^2 + m_e\,\Omega_e^2$$

$$\Rightarrow \qquad = m_i\frac{e^2B_0^2}{m_i^2c^2} + m_e\frac{e^2B_0^2}{m_e^2c^2}$$

$$\Rightarrow \frac{e^2B_0^2}{c^2}\left(\frac{1}{m_i}+\frac{1}{m_e}\right)$$

$$= \frac{e^2B_0^2}{c^2}\left(\frac{m_e+m_i}{m_im_e}\right)$$

$$\Rightarrow \omega^2 = \frac{e^2B_0^2}{c^2}\cdot\frac{1}{m_im_e} = \left(\frac{eB_0}{m_ic}\right)\cdot\left(\frac{eB_0}{m_ec}\right)$$

$$\therefore\ \omega = (\Omega_i\Omega_e)^{\frac{1}{2}} = \omega_l$$

This is the expression for lower hybrid frequency.

Electromagnetic waves in an electron plasma in absence of $\vec{B}$-field.

To derive the dispersion relation for electromagnetic waves in an electron plasma in absence of $\vec{B}$-field. Let us assume the following assumptions –

i) There is no thermal motion that $KT_e = 0$

ii) The ions are fixed in space in a uniform distribution

iii) The plasma is infinite in extent.

iv) The electron oscillations occur only in the x-direction

v) The plasma is neutral at rest
$$v_0 = 0 \quad , \quad E_0 = 0$$

from the above set of assumptions we have the following electron eqns of motion —

$$\frac{\partial \vec{v}_e}{\partial t} + (\vec{v}_e \cdot \vec{\nabla})\vec{v}_e = -\frac{e\vec{E}}{m_e} - \frac{e}{m_e c}(\vec{v}_e \times \vec{B}) \quad \text{—— (1)}$$

also we have the following equations from Maxwell's equations

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \quad \text{—— (2)}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J} \qquad \vec{J} = -en_e\vec{v}_e \quad \text{—— (3)}$$

$$\left.\begin{aligned} \vec{v}_e &= \vec{v}_{e1} \\ \vec{E} &= \vec{E}_1 \\ \vec{B} &= \vec{B}_1 \\ n_e &= n_0 + n_1 \end{aligned}\right\} \text{(A)} \qquad n_0, n_0 \ldots$$

using (A) in (1) to (3)

we get

$$\frac{\partial \vec{v}_{e1}}{\partial t} = -\frac{e\vec{E}_1}{m_e} \quad \text{--- (4)}$$

$$\vec{\nabla}\times\vec{E}_1 = -\frac{1}{c}\frac{\partial \vec{B}_1}{\partial t} \quad \text{--- (5)}$$

$$\vec{\nabla}\times\vec{B}_1 = \frac{1}{c}\frac{\partial \vec{E}_1}{\partial t} - \frac{4\pi}{c} e n_0 \vec{v}_{e1} \quad \text{--- (6)}$$

now ~~in~~ writing componentwise the equations (4) to (6) becomes —

$$\frac{\partial v_{e1y}}{\partial t} = -\frac{eE_{1y}}{m_e} \quad \text{--- (7)}$$

$$\frac{\partial v_{e1z}}{\partial t} = -\frac{eE_{1z}}{m_e} \quad \text{--- (8)}$$

$$\frac{\partial E_{1y}}{\partial x} = -\frac{1}{c}\frac{\partial B_{1z}}{\partial t} \quad \text{--- (9)}$$

$$\frac{\partial E_{1z}}{\partial x} = \frac{1}{c}\frac{\partial B_{1y}}{\partial t} \quad \text{--- (10)}$$

$$\frac{\partial B_{1y}}{\partial x} = \frac{1}{c}\frac{\partial E_{1z}}{\partial t} - \frac{4\pi e n_0}{c} v_{e1z} \quad \text{--- (11)}$$

$$\frac{\partial B_{1z}}{\partial x} = -\frac{1}{c}\frac{\partial E_{1y}}{\partial t} + \frac{4\pi e n_0}{c} v_{e1y} \quad \text{--- (12)}$$

$$\left[\because \vec{E}_1 = E_{1y}\hat{y} + E_{1z}\hat{z}, \quad \vec{B}_1 = B_{1y}\hat{y} + B_{1z}\hat{z}\right]$$

using the plane wave solution time derivative can be replace by $-i\omega$ and $\frac{\partial}{\partial x}$ can be replace by ik

Then the above set of eqns becomes:—

$-i\omega v_{e1y} = -\frac{eE_{1y}}{m_e}$

$i\omega v_{e1y} = \frac{eE_{1y}}{m_e}$ —— (13)

$-i\omega v_{e1z} = -\frac{eE_{1z}}{m_e}$

$\Rightarrow$ $i\omega v_{e1z} = \frac{eE_{1z}}{m_e}$ —— (14)

$ikE_{1y} = \frac{+1}{c} i\omega B_{1z}$ —— (15)

$ikE_{1z} = -\frac{1}{c} i\omega B_{1y}$ —— (16)

$ikB_{1y} = \frac{+1}{c} i\omega E_{1z} - \frac{4\pi e n_0}{c} v_{e1z}$ —— (17)

$ikB_{1z} = +\frac{1}{c} i\omega E_{1y} + \frac{4\pi e n_0}{c} v_{e1y}$ —— (18)

eqns (13), (15) & (18) can be written as

$i\omega v_{e1y} - \frac{e}{m_e} E_{1y} + 0 \cdot B_{1z} = 0$

$0 \cdot v_{e1y} + ikE_{1y} - \frac{i\omega}{c} B_{1z} = 0$

$\frac{4\pi e n_0}{c} v_{e1y} + \frac{1}{c} i\omega E_{1y} - ikB_{1z} = 0$

elliminating v_{e1y}, E_{1y} & B_{1z} we get

$$\begin{vmatrix} i\omega & -\frac{e}{m_e} & 0 \\ 0 & ik & -\frac{i\omega}{c} \\ \frac{4\pi e n_0}{c} & \frac{i\omega}{c} & -ik \end{vmatrix} = 0$$

$\Rightarrow$ $i\omega\left[+k^2 - \frac{\omega^2}{c^2}\right] + \frac{e}{m_e}\left[+\frac{4i\omega\pi e n_0}{c^2}\right] = 0$

$$k^2c^2 - \omega^2 + \frac{e}{m_e} \frac{4\pi i \omega e n_0}{c^2} \times \frac{c^2 i}{\omega} = 0$$

$$\Rightarrow k^2c^2 - \omega^2 + \frac{4\pi e^2 n_0}{m_e} = 0$$

$$\Rightarrow k^2c^2 - \omega^2 + \omega_{pe}^2 = 0$$

$$\Rightarrow \boxed{\omega^2 - \omega_{pe}^2 = k^2c^2} \quad \text{(B)} \qquad \left[\because \omega_{pe} = \sqrt{\frac{4\pi e^2 n_0}{m_e}}\right.$$

electron plasma frequency

Which is recognised dispersion relations.

Now the phase velocity of light wave in plasma can be express as —

$$v_\phi^2 = \frac{\omega^2}{k^2}$$

$$= \frac{\omega_{pe}^2 + k^2c^2}{k^2}$$

$$= c^2 + \frac{\omega_{pe}^2}{k^2} > c^2$$

So phase velocity of light is greater than the velocity of light

The group velocity is given by

$$v_g = \frac{d\omega}{dk}$$

from (B)

$$\omega^2 = \omega_{pe}^2 + k^2c^2$$

$$\Rightarrow 2\omega \frac{d\omega}{dk} = 2kc^2$$

$$\Rightarrow \frac{d\omega}{dk} = \frac{\cancel{\hbar} k c^2}{\omega} = \frac{c^2}{\omega/k} = \frac{c^2}{V_\phi}$$

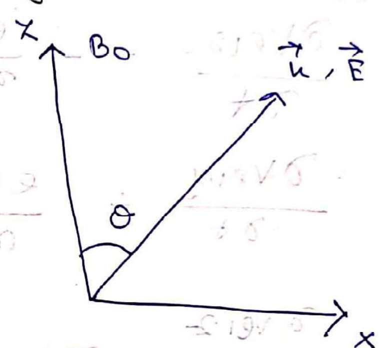
$$V_g = \frac{d\omega}{dk} = \frac{c^2}{V_\phi}$$

Since $V_\phi > c$, So $V_g < c$ i.e group velocity can not exceed the velocity of light.

* V.V.I

Dispersion relation for electrostatic electron waves propagating at any arbitrary angle θ related to magnetic field $\vec{B_0}$.

To Find the dispersion relation 1st write the electrons equation of motion, eqn of continuity and poison's eqns given below —



$$\frac{\partial \vec{v_e}}{\partial t} + (\vec{v_e}\cdot\vec{\nabla})\vec{v_e} = -\frac{e}{m_e}\vec{E} - \frac{e}{m_e c}(\vec{v_e}\times\vec{B}) \quad (1)$$

$$\frac{\partial n_e}{\partial t} + \vec{\nabla}\cdot(n_e\vec{v_e}) = 0 \quad (2)$$

$$\vec{\nabla}\cdot\vec{E} = -4\pi e n_e \quad (3)$$

in the present case we have

$\vec{V_0} = 0$ $\vec{E_0} = 0$ $\vec{B_1} = 0$

therefore we have $\vec{v} = \vec{v_1}$, $\vec{E} = \vec{E_1}$ $\vec{B} = \vec{B_0}$

we have given that $n = n_0 + n_1$ the electrostatic electron waves propagating at an arbitrary angle θ related to $\vec{B_0}$. We take $\vec{B_0}$ along the axis of z. we choose x-axis, such that $\vec{k}, \vec{E}$ lieing (x-z) plane making an angle θ with z-axis. Thus we have

$$E_x = E_1\sin\theta, \quad E_z = E_1\cos\theta, \quad E_y = 0$$

$k_x = k\sin\theta \quad k_z = k\cos\theta \quad k_y = 0$

now linearizing eqn (1), (2) & (3) we get

$$\frac{\partial \vec{v}_{e1}}{\partial t} = -\frac{e}{m_e}\vec{E}_1 - \frac{e}{m_e c}\left(\vec{v}_{e1}\times\vec{B}_0\right) \quad \text{--- (4)}$$

$$\frac{\partial n_{e1}}{\partial t} + \nabla\cdot\left(n_0\vec{v}_{e1}\right) = 0 \quad \text{--- (5)}$$

$$\vec{\nabla}\cdot\vec{E}_1 = -4\pi e n_{e1} \quad \text{--- (6)}$$

now we write component wise the equation (4) to (6) we get —

$$\frac{\partial v_{e1x}}{\partial t} = -\frac{e}{m_e}E_1\sin\theta - \frac{eB_0}{m_e c}v_{e1y}$$

$$\frac{\partial v_{e1y}}{\partial t} = \frac{eB_0}{m_e c}v_{e1x}$$

$$\frac{\partial v_{e1z}}{\partial t} = -\frac{e}{m_e}E_1\cos\theta$$

$$\frac{\partial n_{e1}}{\partial t} + n_0\left(\frac{\partial v_{e1x}}{\partial x} + \frac{\partial v_{e1y}}{\partial y} + \frac{\partial v_{e1z}}{\partial z}\right) = 0$$

$$\frac{\partial E_1\sin\theta}{\partial x} + \frac{\partial E_1\cos\theta}{\partial z} = -4\pi e n_{e1}$$

following plasma wave solutions above system of equations we get

$$-i\omega v_{e1x} = -\frac{e}{m_e}E_1\sin\theta - \omega_e v_{e1y} \quad \text{--- (7)}$$

$$-i\omega v_{e1y} = \omega_e v_{e1x} \quad \text{--- (8)}$$

$$-i\omega v_{e1z} = -\frac{e}{m_e c}E_1\cos\theta \quad \text{--- (9)}$$

$$-i\omega n_{e1} + n_0 ik\sin\theta\, v_{e1x} + ikn_0 v_{e1z}\cos\theta = 0 \quad \text{--- (10)}$$

$$ikE_1\sin^2\theta + ik\cos^2\theta\, E_1 = -4\pi e n_{e1} \quad \text{--- (11)}$$